

This chapter is an introduction to state variables and state equations as they apply in circuit analysis. The state transition matrix is defined, and the state space-to-transfer function equivalence is presented. Several examples are given to illustrate their application.

5.1 Expressing Differential Equations in State Equation Form

As we know, when we apply KCL or KVL in networks that contain energy-storing devices, we obtain integro-differential equations. Also, when a network contains just one such device (capacitor or inductor), it is said to be a *first order circuit*. If it contains two such devices, it is said to be *second-order circuit*, and so on. Thus, a first order linear, time-invariant circuit can be described by a differential equation of the form

$$a_1 \frac{dy}{dt} + a_0 y(t) = x(t) \quad (5.1)$$

A second order circuit can be described by a second-order differential equation of the same form as (5.1) where the highest order is a second derivative.

An *nth-order* differential equation can be resolved to *n* first-order simultaneous differential equations with a set of auxiliary variables called *state variables*. The resulting first-order differential equations are called *state space equations*, or simply *state equations*. These equations can be obtained either from the *nth-order* differential equation, or directly from the network, provided that the state variables are chosen appropriately. The state variable method offers the advantage that it can also be used with non-linear and time-varying devices. However, our discussion will be limited to linear, time-invariant circuits.

State equations can also be solved with numerical methods such as Taylor series and Runge-Kutta methods, but these will not be discussed in this text*. The state variable method is best illustrated through several examples presented in this chapter.

Example 5.1

A series *RLC* circuit with excitation

$$v_S(t) = e^{j\omega t} \quad (5.2)$$

* These are discussed in “Numerical Analysis using MATLAB and Spreadsheets” ISBN 0-9709511-1-6.

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is described by the integro-differential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^t i dt = e^{j\omega t} \quad (5.3)$$

Differentiating both sides and dividing by L we get

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}j\omega e^{j\omega t} \quad (5.4)$$

or

$$\frac{d^2 i}{dt^2} = -\frac{R}{L}\frac{di}{dt} - \frac{1}{LC}i + \frac{1}{L}j\omega e^{j\omega t} \quad (5.5)$$

Next, we define two state variables x_1 and x_2 such that

$$x_1 = i \quad (5.6)$$

and

$$x_2 = \frac{di}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \quad (5.7)$$

Then,

$$\dot{x}_2 = d^2 i / dt^2 \quad (5.8)$$

where $\dot{x}_k$ denotes the derivative of the state variable x_k .

From (5.5) through (5.8), we obtain the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{L}j\omega e^{j\omega t} \end{aligned} \quad (5.9)$$

It is convenient and customary to express the state equations in matrix^{*} form. Thus, we write the state equations of (5.9) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L}j\omega e^{j\omega t} \end{bmatrix} u \quad (5.10)$$

* For a review of matrix theory, please refer to Appendix C.

We usually write (5.10) in a compact form as

$$\dot{x} = Ax + bu \quad (5.11)$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{1}{L} j\omega e^{j\omega t} \end{bmatrix}, \text{ and } u = \text{any input} \quad (5.12)$$

The output $y(t)$ is expressed by the state equation

$$y = Cx + du \quad (5.13)$$

where C is another matrix, and d is a column vector. Therefore, the state representation of a network can be described by the pair of the of the state space equations

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + du \end{aligned}$$

(5.14)

The state space equations of (5.14) can be realized with the block diagram of Figure 5.1.

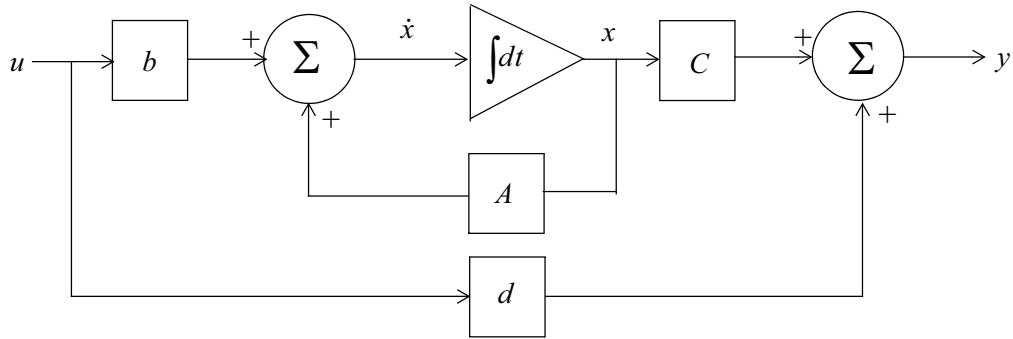


Figure 5.1. Block diagram for the realization of the state equations of (5.14)

We will learn how to solve the matrix equations of (5.14) in the subsequent sections.

Example 5.2

A fourth-order network is described by the differential equation

$$\frac{d^4 y}{dt^4} + a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = u(t) \quad (5.15)$$

where $y(t)$ is the output representing the voltage or current of the network, and $u(t)$ is any input. Express (5.15) as a set of state equations.

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Solution:

The differential equation of (5.15) is of fourth-order; therefore, we must define four state variables that will be used with the resulting four first-order state equations.

We denote the state variables as x_1, x_2, x_3 , and x_4 , and we relate them to the terms of the given differential equation as

$$x_1 = y(t) \quad x_2 = \frac{dy}{dt} \quad x_3 = \frac{d^2y}{dt^2} \quad x_4 = \frac{d^3y}{dt^3} \quad (5.16)$$

We observe that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \frac{d^4y}{dt^4} &= \dot{x}_4 = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u(t) \end{aligned} \quad (5.17)$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (5.18)$$

In compact form, (5.18) is written as

$$\dot{x} = Ax + bu \quad (5.19)$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = u(t)$$

We can also obtain the state equations directly from given circuits. We choose the state variables to represent inductor currents and capacitor voltages. In other words, we assign state variables to energy storing devices. The examples that follow illustrate the procedure.

Example 5.3

Write state equation(s) for the circuit of Figure 5.2, given that $v_C(0^-) = 0$.

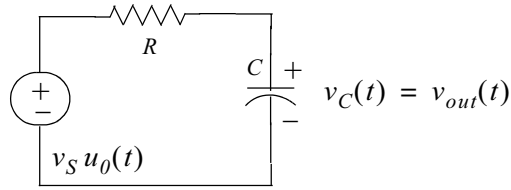


Figure 5.2. Circuit for Example 5.3

Solution:

This circuit contains only one energy-storing device, the capacitor. Therefore, we need only one state variable. We choose the state variable to denote the voltage across the capacitor as shown in Figure 5.3. The output is defined as the voltage across the capacitor.

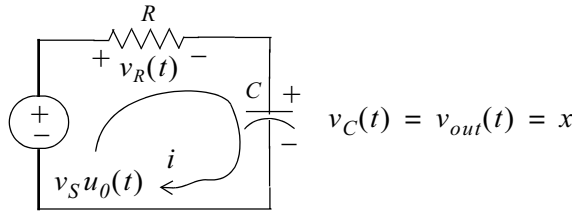


Figure 5.3. Circuit for Example 5.3 with state variable x assigned to it

For this series circuit,

$$i_R = i = i_C = C \frac{dv_C}{dt} = C\dot{x}$$

and

$$v_R(t) = Ri = RC\dot{x}$$

By KVL,

$$v_R(t) + v_C(t) = v_S u_0(t)$$

or

$$RC\dot{x} + x = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned}\dot{x} &= -\frac{1}{RC}x + v_S u_0(t) \\ y &= x\end{aligned}\tag{5.20}$$

Example 5.4

Write state equation(s) for the circuit of Figure 5.4 assuming $i_L(0^-) = 0$, and the output y is defined as $y = i(t)$.

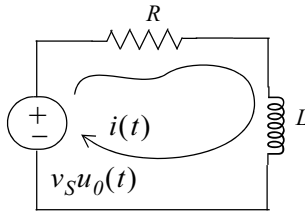


Figure 5.4. Circuit for Example 5.4

Solution:

This circuit contains only one energy-storing device, the inductor; therefore, we need only one state variable. We choose the state variable to denote the current through the inductor as shown in Figure 5.5.

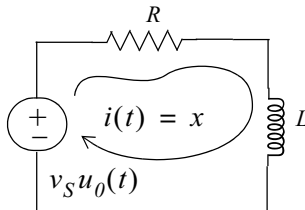


Figure 5.5. Circuit for Example 5.4 with state variable x assigned to it

By KVL,

$$v_R + v_L = v_S u_0(t)$$

or

$$Ri + L \frac{di}{dt} = v_S u_0(t)$$

or

$$Rx + L\dot{x} = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned}\dot{x} &= -\frac{R}{L}x + \frac{1}{L}v_S u_0(t) \\ y &= x\end{aligned}\tag{5.21}$$

5.2 Solution of Single State Equations

If a circuit contains only one energy-storing device, the state equations are written as

$$\begin{aligned}\dot{x} &= \alpha x + \beta u \\ y &= k_1 x + k_2 u\end{aligned}\tag{5.22}$$

where α , β , k_1 , and k_2 are scalar constants, and the initial condition, if non-zero, is denoted as

$$x_0 = x(t_0)\tag{5.23}$$

We will now prove that the solution of the first state equation in (5.22) is

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha\tau} \beta u(\tau) d\tau\tag{5.24}$$

Proof:

First, we must show that (5.24) satisfies the initial condition of (5.23). This is done by substitution of $t = t_0$ in (5.24). Then,

$$x(t_0) = e^{\alpha(t_0-t_0)}x_0 + e^{\alpha t_0} \int_{t_0}^{t_0} e^{-\alpha\tau} \beta u(\tau) d\tau\tag{5.25}$$

The first term in the right side of (5.25) reduces to x_0 since

$$e^{\alpha(t_0-t_0)}x_0 = e^0 x_0 = x_0\tag{5.26}$$

The second term of (5.25) is zero since the upper and lower limits of integration are the same.

Therefore, (5.25) reduces to $x(t_0) = x_0$ and thus the initial condition is satisfied.

Next, we must prove that (5.24) satisfies also the first equation in (5.22). To prove this, we differentiate (5.24) with respect to t and we get

$$\dot{x}(t) = \frac{d}{dt}(e^{\alpha(t-t_0)}x_0) + \frac{d}{dt}\left\{e^{\alpha t} \int_{t_0}^t e^{-\alpha\tau} \beta u(\tau) d\tau\right\}$$

or

$$\begin{aligned}\dot{x}(t) &= \alpha e^{\alpha(t-t_0)}x_0 + \alpha e^{\alpha t} \int_{t_0}^t e^{-\alpha\tau} \beta u(\tau) d\tau + e^{\alpha t} [e^{-\alpha\tau} \beta u(\tau)] \Big|_{\tau=t} \\ &= \alpha \left[e^{\alpha(t-t_0)}x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha\tau} \beta u(\tau) d\tau \right] + e^{\alpha t} e^{-\alpha t} \beta u(t)\end{aligned}$$

or

$$\dot{x}(t) = \alpha \left[e^{\alpha(t-t_0)} x_0 + \int_{t_0}^t e^{\alpha(t-\tau)} \beta u(\tau) d\tau \right] + \beta u(t) \quad (5.27)$$

We observe that the bracketed terms of (5.27) are the same as the right side of the assumed solution of (5.24). Therefore,

$$\dot{x} = \alpha x + \beta u$$

and this is the same as the first equation of (5.22).

In summary, if α and β are scalar constants, the solution of

$$\dot{x} = \alpha x + \beta u \quad (5.28)$$

with initial condition

$$x_0 = x(t_0) \quad (5.29)$$

is obtained from the relation

$$x(t) = e^{\alpha(t-t_0)} x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha \tau} \beta u(\tau) d\tau \quad (5.30)$$

Example 5.5

Use (5.28) through (5.30) to find the capacitor voltage $v_c(t)$ of the circuit of Figure 5.6 for $t > 0$, given that the initial condition is $v_c(0^-) = 1 \text{ V}$

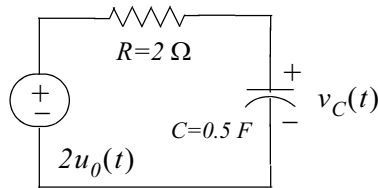


Figure 5.6. Circuit for Example 5.5

Solution:

From (5.20) of Example 5.3,

$$\dot{x} = -\frac{I}{RC}x + v_S u_0(t)$$

and by comparison with (5.28),

$$\alpha = -\frac{I}{RC} = \frac{-1}{2 \times 0.5} = -1$$

and

$$\beta = 2$$

Then, from (5.30),

$$\begin{aligned} x(t) &= e^{\alpha(t-t_0)} x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha \tau} \beta u(\tau) d\tau = e^{-1(t-0)} I + e^{-t} \int_0^t e^{\tau} 2u(\tau) d\tau \\ &= e^{-t} + 2e^{-t} \int_0^t e^{\tau} d\tau = e^{-t} + 2e^{-t} [e^{\tau}]_0^t = e^{-t} + 2e^{-t}(e^t - 1) \end{aligned}$$

or

$$v_C(t) = x(t) = (2 - e^{-t})u_0(t) \quad (5.31)$$

If we assume that the output y is the capacitor voltage, the output state equation is

$$y(t) = x(t) = (2 - e^{-t})u_0(t) \quad (5.32)$$

5.3 The State Transition Matrix

In Section 5.1 we defined the state equations pair

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + du \end{aligned} \quad (5.33)$$

where for two or more simultaneous differential equations, A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows. In this section we will introduce the *state transition matrix* e^{At} , and we will prove that the solution of the matrix differential equation

$$\dot{x} = Ax + bu \quad (5.34)$$

with initial conditions

$$x(t_0) = x_0 \quad (5.35)$$

is obtained from the relation

$$x(t) = e^{A(t-t_0)} x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau \quad (5.36)$$

Proof:

Let A be any $n \times n$ matrix whose elements are constants. Then, another $n \times n$ matrix denoted as $\varphi(t)$, is said to be the state transition matrix of (5.34), if it is related to the matrix A as the matrix power series

$$\varphi(t) \equiv e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots + \frac{1}{n!} A^n t^n$$

(5.37)

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where I is the $n \times n$ identity matrix.

From (5.37), we find that

$$\varphi(0) = e^{A0} = I + A0 + \dots = I \quad (5.38)$$

Differentiation of (5.37) with respect to t yields

$$\varphi'(t) = \frac{d}{dt}e^{At} = 0 + A \cdot I + A^2t + \dots = A + A^2t + \dots \quad (5.39)$$

and by comparison with (5.37) we get

$$\frac{d}{dt}e^{At} = Ae^{At} \quad (5.40)$$

To prove that (5.36) is the solution of (5.34), we must prove that it satisfies both the initial condition and the matrix differential equation. The initial condition is satisfied from the relation

$$x(t_0) = e^{A(t_0-t_0)}x_0 + e^{At_0} \int_{t_0}^{t_0} e^{-A\tau} bu(\tau) d\tau = e^{A0}x_0 + 0 = Ix_0 = x_0 \quad (5.41)$$

where we have used (5.38) for the initial condition. The integral is zero since the upper and lower limits of integration are the same.

To prove that (5.34) is also satisfied, we differentiate the assumed solution

$$x(t) = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau$$

with respect to t and we use (5.40), that is,

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Then,

$$\dot{x}(t) = Ae^{A(t-t_0)}x_0 + Ae^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau + e^{At} e^{-At} bu(t)$$

or

$$\dot{x}(t) = A \left[e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau \right] + e^{At} e^{-At} bu(t) \quad (5.42)$$

We recognize the bracketed terms in (5.42) as $x(t)$, and the last term as $bu(t)$. Thus, the expression (5.42) reduces to

$$\dot{x}(t) = Ax + bu$$

In summary, if A is an $n \times n$ matrix whose elements are constants, $n \geq 2$, and b is a column vector with n elements, the solution of

$$\dot{x}(t) = Ax + bu \quad (5.43)$$

with initial condition

$$x_0 = x(t_0) \quad (5.44)$$

is

$$x(t) = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau \quad (5.45)$$

Therefore, the solution of second or higher order circuits using the state variable method, entails the computation of the state transition matrix e^{At} , and integration of (5.45).

5.4 Computation of the State Transition Matrix e^{At}

Let A be an $n \times n$ matrix, and I be the $n \times n$ identity matrix. By definition, the *eigenvalues* λ_i , $i = 1, 2, \dots, n$ of A are the roots of the n th order polynomial

$$\det[A - \lambda I] = 0 \quad (5.46)$$

We recall that expansion of a determinant produces a polynomial. The roots of the polynomial of (5.46) can be real (unequal or equal), or complex numbers.

Evaluation of the state transition matrix e^{At} is based on the *Cayley-Hamilton theorem*. This theorem states that a matrix can be expressed as an $(n-1)$ th degree polynomial in terms of the matrix A as

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1} \quad (5.47)$$

where the coefficients a_i are functions of the eigenvalues λ .

We accept (5.47) without proving it. The proof can be found in Linear Algebra and Matrix Theory textbooks.

Since the coefficients a_i are functions of the eigenvalues λ , we must consider the following cases:

Case I: Distinct Eigenvalues (Real or Complex)

If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \dots \neq \lambda_n$, that is, if all eigenvalues of a given matrix A are distinct, the coefficients a_i are found from the simultaneous solution of the following system of equations:

$$\begin{aligned} a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1 \lambda_2 + a_2 \lambda_2^2 + \dots + a_{n-1} \lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\dots \\ a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \dots + a_{n-1} \lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned} \quad (5.48)$$

Example 5.6

Compute the state transition matrix e^{At} given that $A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

Solution:

We must first find the eigenvalues λ of the given matrix A . These are found from the expansion of

$$\det[A - \lambda I] = 0$$

For this example,

$$\begin{aligned} \det[A - \lambda I] &= \det \left\{ \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \det \begin{bmatrix} -2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = 0 \\ &= (-2-\lambda)(-1-\lambda) = 0 \end{aligned}$$

or

$$(\lambda + 1)(\lambda + 2) = 0$$

Therefore,

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2 \quad (5.49)$$

Next, we must find the coefficients a_i of (5.47). Since A is a 2×2 matrix, we only need to consider the first two terms of that relation, that is,

$$e^{At} = a_0 I + a_1 A \quad (5.50)$$

The coefficients a_0 and a_1 are found from (5.48). For this example,

$$a_0 + a_1 \lambda_1 = e^{\lambda_1 t}$$

$$a_0 + a_1 \lambda_2 = e^{\lambda_2 t}$$

or

$$a_0 + a_1(-1) = e^{-t}$$

$$a_0 + a_1(-2) = e^{-2t} \quad (5.51)$$

Simultaneous solution of (5.51) yields

$$\begin{aligned} a_0 &= 2e^{-t} - e^{-2t} \\ a_1 &= e^{-t} - e^{-2t} \end{aligned} \quad (5.52)$$

and by substitution into (5.50),

$$e^{At} = (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} e^{-2t} & e^{-t} - e^{-2t} \\ 0 & e^{-t} \end{bmatrix} \quad (5.53)$$

In summary, we compute the state transition matrix e^{At} for a given matrix A using the following procedure:

1. We find the eigenvalues λ from $\det[A - \lambda I] = 0$. We can write $[A - \lambda I]$ at once by subtracting λ from each of the main diagonal elements of A . If the dimension of A is a 2×2 matrix, it will yield two eigenvalues; if it is a 3×3 matrix, it will yield three eigenvalues, and so on. If the eigenvalues are distinct, we perform steps 2 through 4; otherwise we refer to Case II below.
2. If the dimension of A is a 2×2 matrix, we use only the first 2 terms of the right side of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1} \quad (5.54)$$

If A matrix is a 3×3 matrix, we use the first 3 terms, and so on.

3. We obtain the a_i coefficients from

$$\begin{aligned} a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1 \lambda_2 + a_2 \lambda_2^2 + \dots + a_{n-1} \lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\dots \\ a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \dots + a_{n-1} \lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned}$$

We use as many equations as the number of the eigenvalues, and we solve for the coefficients a_i .

4. We substitute the a_i coefficients into the state transition matrix of (5.54), and we simplify.

Example 5.7

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \quad (5.55)$$

Solution:

1. We first compute the eigenvalues from $\det[A - \lambda I] = 0$. We obtain $[A - \lambda I]$ at once, by subtracting λ from each of the main diagonal elements of A . Then,

$$\det[A - \lambda I] = \det \begin{bmatrix} 5 - \lambda & 7 & -5 \\ 0 & 4 - \lambda & -1 \\ 2 & 8 & -3 - \lambda \end{bmatrix} = 0 \quad (5.56)$$

and expansion of this determinant yields the polynomial

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad (5.57)$$

We will use MATLAB **roots(p)** function to obtain the roots of (5.57).

```
p=[1 -6 11 -6]; r=roots(p); fprintf(' \n'); fprintf('lambda1 = %5.2f \t', r(1));...  
fprintf('lambda2 = %5.2f \t', r(2)); fprintf('lambda3 = %5.2f', r(3))
```

```
lambda1 = 3.00    lambda2 = 2.00    lambda3 = 1.00
```

and thus the eigenvalues are

$$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3 \quad (5.58)$$

2. Since A is a 3×3 matrix, we need to use the first 3 terms of (5.54), that is,

$$e^{At} = a_0 I + a_1 A + a_2 A^2 \quad (5.59)$$

3. We obtain the coefficients a_0 , a_1 , and a_2 from

$$a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 = e^{\lambda_1 t}$$

$$a_0 + a_1 \lambda_2 + a_2 \lambda_2^2 = e^{\lambda_2 t}$$

$$a_0 + a_1 \lambda_3 + a_2 \lambda_3^2 = e^{\lambda_3 t}$$

or

$$\begin{aligned}
 a_0 + a_1 + a_2 &= e^t \\
 a_0 + 2a_1 + 4a_2 &= e^{2t} \\
 a_0 + 3a_1 + 9a_2 &= e^{3t}
 \end{aligned}
 \tag{5.60}$$

We will use the following MATLAB code for the solution of (5.60).

```
B=sym('[1 1 1; 1 2 4; 1 3 9]'); b=sym('[exp(t); exp(2*t); exp(3*t)]'); a=B\b; fprintf('\n');...
disp('a0 = '); disp(a(1)); disp('a1 = '); disp(a(2)); disp('a2 = '); disp(a(3))
```

```
a0 =
3*exp(t) - 3*exp(2*t) + exp(3*t)
a1 =
-5/2*exp(t) + 4*exp(2*t) - 3/2*exp(3*t)
a2 =
1/2*exp(t) - exp(2*t) + 1/2*exp(3*t)
```

Thus,

$$\begin{aligned}
 a_0 &= 3e^t - 3e^{2t} + e^{3t} \\
 a_1 &= -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t} \\
 a_2 &= \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}
 \end{aligned}
 \tag{5.61}$$

4. We also use MATLAB to perform the substitution into the state transition matrix, and to perform the matrix multiplications. The code is shown below.

```
syms t; a0 = 3*exp(t)+exp(3*t)-3*exp(2*t); a1 = -5/2*exp(t)-3/2*exp(3*t)+4*exp(2*t);...
a2 = 1/2*exp(t)+1/2*exp(3*t)-exp(2*t);...
A = [5 7 -5; 0 4 -1; 2 8 -3]; eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
[ -2*exp(t)+2*exp(2*t)+exp(3*t), -6*exp(t)+5*exp(2*t)+exp(3*t), 4*exp(t)-3*exp(2*t)-exp(3*t) ]
[ -exp(t)+2*exp(2*t)-exp(3*t), -3*exp(t)+5*exp(2*t)-exp(3*t), 2*exp(t)-3*exp(2*t)+exp(3*t) ]
[ -3*exp(t)+4*exp(2*t)-exp(3*t), -9*exp(t)+10*exp(2*t)-exp(3*t), 6*exp(t)-6*exp(2*t)+exp(3*t) ]
```

Thus,

$$e^{At} = \begin{bmatrix} -2e^t + 2e^{2t} + e^{3t} & -6e^t + 5e^{2t} + e^{3t} & 4e^t - 3e^{2t} - e^{3t} \\ -e^t + 2e^{2t} - e^{3t} & -3e^t + 5e^{2t} - e^{3t} & 2e^t - 3e^{2t} + e^{3t} \\ -3e^t + 4e^{2t} - e^{3t} & -9e^t + 10e^{2t} - e^{3t} & 6e^t - 6e^{2t} + e^{3t} \end{bmatrix}$$

Case II: Multiple Eigenvalues

In this case, we will assume that the polynomial of

$$\det[A - \lambda I] = 0
 \tag{5.62}$$

Chapter 5 State Variables and State Equations

has n roots, and m of these roots are equal. In other words, the roots are

$$\lambda_1 = \lambda_2 = \lambda_3 \dots = \lambda_m, \lambda_{m+1}, \lambda_n \quad (5.63)$$

The coefficients a_i of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1} \quad (5.64)$$

are found from the simultaneous solution of the system of equations of (5.65) below.

$$\begin{aligned} a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1} &= e^{\lambda_1 t} \\ \frac{d}{d\lambda_1} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) &= \frac{d}{d\lambda_1} e^{\lambda_1 t} \\ \frac{d^2}{d\lambda_1^2} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2} e^{\lambda_1 t} \\ &\dots \\ \frac{d^{m-1}}{d\lambda_1^{m-1}} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}} e^{\lambda_1 t} \\ a_0 + a_1 \lambda_{m+1} + a_2 \lambda_{m+1}^2 + \dots + a_{n-1} \lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\ &\dots \\ a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \dots + a_{n-1} \lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned} \quad (5.65)$$

Example 5.8

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Solution:

1. We first find the eigenvalues λ of the matrix A and these are found from the polynomial of $\det[A - \lambda I] = 0$. For this example,

$$\begin{aligned} \det[A - \lambda I] &= \det \begin{bmatrix} -1 - \lambda & 0 \\ 2 & -1 - \lambda \end{bmatrix} = 0 \\ &= (-1 - \lambda)(-1 - \lambda) = 0 \\ &= (\lambda + 1)^2 = 0 \end{aligned}$$

and thus,

$$\lambda_1 = \lambda_2 = -I$$

2. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \quad (5.66)$$

3. We find a_0 and a_1 from (5.65). For this example,

$$\begin{aligned} a_0 + a_1 \lambda_1 &= e^{\lambda_1 t} \\ \frac{d}{d\lambda_1}(a_0 + a_1 \lambda_1) &= \frac{d}{d\lambda_1} e^{\lambda_1 t} \end{aligned}$$

or

$$\begin{aligned} a_0 + a_1 \lambda_1 &= e^{\lambda_1 t} \\ a_1 &= t e^{\lambda_1 t} \end{aligned}$$

and by substitution with $\lambda_1 = \lambda_2 = -I$, we get

$$\begin{aligned} a_0 - a_1 &= e^{-t} \\ a_1 &= t e^{-t} \end{aligned}$$

Simultaneous solution of the last two equations yields

$$\begin{aligned} a_0 &= e^{-t} + t e^{-t} \\ a_1 &= t e^{-t} \end{aligned} \quad (5.67)$$

4. By substitution of (5.67) into (5.66), we get

$$e^{At} = (e^{-t} + t e^{-t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t e^{-t} \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2t e^{-t} & e^{-t} \end{bmatrix} \quad (5.68)$$

We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix. To find out how it is used, we invoke the **help eig** command.

Chapter 5 State Variables and State Equations

We will first use MATLAB to verify the values of the eigenvalues found in Examples 5.6 through 5.8, and we will briefly discuss eigenvectors on the next section.

For Example 5.6

```
A = [-2 1; 0 -1]; lambda=eig(A)
```

```
lambda =  
    -2  
    -1
```

For Example 5.7

```
B = [5 7 -5; 0 4 -1; 2 8 -3]; lambda=eig(B)
```

```
lambda =  
    1.0000  
    3.0000  
    2.0000
```

For Example 5.8

```
C = [-1 0; 2 -1]; lambda=eig(C)
```

```
lambda =  
    -1  
    -1
```

5.5 Eigenvectors

Consider the relation

$$AX = \lambda X \quad (5.69)$$

where A is an $n \times n$ matrix, X is a column vector, and λ is a scalar number. We can express this relation in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad (5.70)$$

We write (5.70) as

$$(A - \lambda I)X = 0 \quad (5.71)$$

Then, (5.71) can be written as

$$\begin{bmatrix} (a_{11}-\lambda)x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & (a_{22}-\lambda)x_2 & \dots & a_{2n}x_n \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 & a_{n2}x_2 & \dots & (a_{nn}-\lambda)x_n \end{bmatrix} = 0 \quad (5.72)$$

The equations of (5.72) will have non-trivial solutions if and only if its determinant is zero^{*}, that is, if

$$\det \begin{bmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{bmatrix} = 0 \quad (5.73)$$

Expansion of the determinant of (5.73) results in a polynomial equation of degree n in λ , and it is called the *characteristic equation*.

We can express (5.73) in a compact form as

$$\det(A-\lambda I) = 0 \quad (5.74)$$

As we know, the roots λ of the characteristic equation are the eigenvalues of the matrix A , and corresponding to each eigenvalue λ , there is a non-trivial solution of the column vector X , i.e., $X \neq 0$. This vector X is called *eigenvector*. Obviously, there is a different eigenvector for each eigenvalue. Eigenvectors are generally expressed as *unit eigenvectors*, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.

In many engineering applications the unit eigenvectors are chosen such that $X \cdot X^T = I$ where X^T is the transpose of the eigenvector X , and I is the identity matrix.

Two vectors X and Y are said to be *orthogonal* if their inner (dot) product is zero. A set of eigenvectors constitutes an *orthonormal basis* if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal. An orthonormal basis can be formed with the *Gram-Schmidt Orthogonalization Procedure*; it is beyond the scope of this chapter to discuss this procedure, and therefore it will not be discussed in this text. It can be found in Linear Algebra and Matrix Theory textbooks.

The example which follows, illustrates the relationships between a matrix A , its eigenvalues, and eigenvectors.

* This is because we want the vector X in (5.71) to be a non-zero vector and the product $(A-\lambda I)X$ to be zero.

Example 5.9

Given the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

- Find the eigenvalues of A
- Find eigenvectors corresponding to each eigenvalue of A
- Form a set of unit eigenvectors using the eigenvectors of part (b).

Solution:

- This is the same matrix as in Example 5.7, where we found the eigenvalues to be

$$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

- We start with

$$AX = \lambda X$$

and we let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then,

$$\begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (5.75)$$

or

$$\begin{bmatrix} 5x_1 & 7x_2 & -5x_3 \\ 0 & 4x_2 & -x_3 \\ 2x_1 & 8x_2 & -3x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} \quad (5.76)$$

Equating corresponding rows and rearranging, we get

$$\begin{bmatrix} (5-\lambda)x_1 & 7x_2 & -5x_3 \\ 0 & (4-\lambda)x_2 & -x_3 \\ 2x_1 & 8x_2 & -(3-\lambda)x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (5.77)$$

For $\lambda = 1$, (5.77) reduces to

$$\begin{aligned} 4x_1 + 7x_2 - 5x_3 &= 0 \\ 3x_2 - x_3 &= 0 \\ 2x_1 + 8x_2 - 4x_3 &= 0 \end{aligned} \quad (5.78)$$

By Crame's rule, or MATLAB, we get the indeterminate values

$$x_1 = 0/0 \quad x_2 = 0/0 \quad x_3 = 0/0 \quad (5.79)$$

Since the unknowns x_1 , x_2 , and x_3 are scalars, we can assume that one of these, say x_2 , is known, and solve x_1 and x_3 in terms of x_2 . Then, we get $x_1 = 2x_2$, and $x_3 = 3x_2$.

Therefore, an eigenvector for $\lambda = 1$ is

$$X_{\lambda=1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 3x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad (5.80)$$

since any eigenvector is a scalar multiple of the last vector in (5.80).

Similarly, for $\lambda = 2$, we get $x_1 = x_2$, and $x_3 = 2x_2$. Then, an eigenvector for $\lambda = 2$ is

$$X_{\lambda=2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad (5.81)$$

Finally, for $\lambda = 3$, we get $x_1 = -x_2$, and $x_3 = x_2$. Then, an eigenvector for $\lambda = 3$ is

$$X_{\lambda=3} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad (5.82)$$

- c. We find the unit eigenvectors by dividing the components of each vector by the square root of the sum of the squares of the components. These are:

$$\sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

The unit eigenvectors are

$$\begin{aligned} \text{Unit } X_{\lambda=1} &= \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} & \text{Unit } X_{\lambda=2} &= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} & \text{Unit } X_{\lambda=3} &= \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \end{aligned} \quad (5.83)$$

We observe that for the first unit eigenvector the sum of the squares is unity, that is,

$$\left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 = \frac{4}{14} + \frac{1}{14} + \frac{9}{14} = 1 \quad (5.84)$$

and the same is true for the other two unit eigenvectors in (5.83).

5.6 Circuit Analysis with State Variables

In this section, we will present two examples to illustrate how the state variable method is used in circuit analysis.

Example 5.10

For the circuit of Figure 5.7, the initial conditions are $i_L(0^-) = 0$, and $v_C(0^-) = 0.5 \text{ V}$. Use the state variable method to compute $i_L(t)$ and $v_C(t)$.

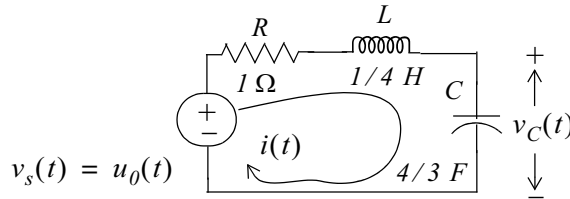


Figure 5.7. Circuit for Example 5.10

Solution:

For this example,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging, yields

$$\frac{1}{4}\frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4 \quad (5.85)$$

Next, we define the state variables $x_1 = i_L$ and $x_2 = v_C$. Then,

$$\dot{x}_1 = \frac{di_L}{dt} \quad (5.86)$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C\frac{dv_C}{dt} = C\dot{x}_2 = \frac{4}{3}\dot{x}_2$$

or

$$\dot{x}_2 = \frac{3}{4}x_1 \quad (5.87)$$

Therefore, from (5.85), (5.86), and (5.87), we get the state equations

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$

$$\dot{x}_2 = \frac{3}{4}x_1$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t) \quad (5.88)$$

We will compute the solution of (5.88) using

$$x(t) = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} b u(\tau) d\tau \quad (5.89)$$

where

$$A = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \quad x_0 = \begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad (5.90)$$

First, we compute the state transition matrix e^{At} . We find the eigenvalues from

$$\det[A - \lambda I] = 0$$

Then,

$$\begin{aligned} \det[A - \lambda I] &= \det \begin{bmatrix} -4 - \lambda & -4 \\ 3/4 & -\lambda \end{bmatrix} = 0 \\ &= (-\lambda)(-4 - \lambda) + 3 = 0 \\ &= \lambda^2 + 4\lambda + 3 = 0 \end{aligned}$$

Therefore,

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -3$$

The next step is to find the coefficients a_i . Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \quad (5.91)$$

The constants a_0 and a_1 are found from

$$\begin{aligned} a_0 + a_1 \lambda_1 &= e^{\lambda_1 t} \\ a_0 + a_1 \lambda_2 &= e^{\lambda_2 t} \end{aligned}$$

and with $\lambda_1 = -1$ and $\lambda_2 = -3$, we get

$$\begin{aligned} a_0 - a_1 &= e^{-t} \\ a_0 - 3a_1 &= e^{-3t} \end{aligned} \quad (5.92)$$

Simultaneous solution of (5.92) yields

$$\begin{aligned} a_0 &= 1.5e^{-t} - 0.5e^{-3t} \\ a_1 &= 0.5e^{-t} - 0.5e^{-3t} \end{aligned} \quad (5.93)$$

We now substitute these values into (5.91), and we get

$$\begin{aligned}
 e^{At} &= (1.5e^{-t} - 0.5e^{-3t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (0.5e^{-t} - 0.5e^{-2t}) \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0 \\ 0 & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} + \begin{bmatrix} -2e^{-t} + 2e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 0 \end{bmatrix}
 \end{aligned}$$

or

$$e^{At} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}$$

The initial conditions vector is the second vector in (5.90); then, the first term of (5.89) becomes

$$e^{At} x_0 = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

or

$$e^{At} x_0 = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} \quad (5.94)$$

We also need to evaluate the integral on the right side of (5.89). From (5.90)

$$b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4$$

and denoting this integral as *Int*, we have

$$Int = \int_{t_0}^t \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} & -2e^{-(t-\tau)} + 2e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} & 1.5e^{-(t-\tau)} - 0.5e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4 d\tau$$

or

$$Int = \int_{t_0}^t \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} \end{bmatrix} 4 d\tau \quad (5.95)$$

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The integration in (5.95) is with respect to τ ; then, integrating the column vector under the integral, we get

$$Int = 4 \left[\begin{array}{c} -0.5e^{-(t-\tau)} + 0.5e^{-3(t-\tau)} \\ 0.375e^{-(t-\tau)} - 0.125e^{-3(t-\tau)} \end{array} \right] \bigg|_{\tau=0}^t$$

or

$$Int = 4 \left[\begin{array}{c} -0.5 + 0.5 \\ 0.375 - 0.125 \end{array} \right] - 4 \left[\begin{array}{c} -0.5e^{-t} + 0.5e^{-3t} \\ 0.375e^{-t} - 0.125e^{-3t} \end{array} \right] = 4 \left[\begin{array}{c} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{array} \right]$$

By substitution of these values, the solution of

$$x(t) = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau$$

is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} + 4 \begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

Then,

$$x_1 = i_L = e^{-t} - e^{-3t} \quad (5.96)$$

and

$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t} \quad (5.97)$$

Other variables of the circuit can now be computed from (5.96) and (5.97). For example, the voltage across the inductor is

$$v_L = L \frac{di_L}{dt} = \frac{1}{4} \frac{d}{dt}(e^{-t} - e^{-3t}) = -\frac{1}{4}e^{-t} + \frac{3}{4}e^{-3t}$$

Example 5.11

A circuit is described by the state equation

$$\dot{x} = Ax + bu \quad (5.98)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \delta(t) \quad (5.99)$$

Compute the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

We compute the eigenvalues from

$$\det[A - \lambda I] = 0$$

For this example,

$$\begin{aligned} \det[A - \lambda I] &= \det \begin{bmatrix} 1 - \lambda & 0 \\ 1 & -1 - \lambda \end{bmatrix} = 0 \\ &= (1 - \lambda)(-1 - \lambda) = 0 \end{aligned}$$

Then,

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -1$$

Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix to find the coefficients a_i , that is,

$$e^{At} = a_0 I + a_1 A \tag{5.100}$$

The constants a_0 and a_1 are found from

$$\begin{aligned} a_0 + a_1 \lambda_1 &= e^{\lambda_1 t} \\ a_0 + a_1 \lambda_2 &= e^{\lambda_2 t} \end{aligned} \tag{5.101}$$

and with $\lambda_1 = 1$ and $\lambda_2 = -1$, we get

$$\begin{aligned} a_0 + a_1 &= e^t \\ a_0 - a_1 &= e^{-t} \end{aligned} \tag{5.102}$$

and simultaneous solution of (5.102) yields

$$\begin{aligned} a_0 &= \frac{e^t + e^{-t}}{2} = \cosh t \\ a_1 &= \frac{e^t - e^{-t}}{2} = \sinh t \end{aligned}$$

By substitution of these values into (5.100), we get

$$e^{At} = \cosh t I + \sinh t A = \cosh t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh t \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cosh t + \sinh t & 0 \\ \sinh t & \cosh t - \sinh t \end{bmatrix} \quad (5.103)$$

The values of the vector $\mathbf{x}$ are found from

$$x(t) = e^{A(t-t_0)} x_0 + e^{At} \int_{t_0}^t e^{-A\tau} b u(\tau) d\tau = e^{At} x_0 + e^{At} \int_0^t e^{-A\tau} b \delta(\tau) d\tau \quad (5.104)$$

Using the sifting property of the delta function we find that (5.104) reduces to

$$\begin{aligned} x(t) &= e^{At} x_0 + e^{At} b = e^{At} (x_0 + b) = e^{At} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = e^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \cosh t + \sinh t & 0 \\ \sinh t & \cosh t - \sinh t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cosh t - \sinh t \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \quad (5.105)$$

5.7 Relationship between State Equations and Laplace Transform

In this section, we will show that the state transition matrix can be computed from the Inverse Laplace transform. We will also show that the transfer function can be found from the coefficient matrices of the state equations.

Consider the state equation

$$\dot{x} = Ax + bu \quad (5.106)$$

Taking the Laplace of both sides of (5.106), we get

$$sX(s) - x(0) = AX(s) + bU(s)$$

or

$$(sI - A)X(s) = x(0) + bU(s) \quad (5.107)$$

Multiplying both sides of (5.107) by $(sI - A)^{-1}$, we get

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} b U(s) \quad (5.108)$$

Comparing (5.108) with

$$x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-A\tau} b u(\tau) d\tau \quad (5.109)$$

we observe that the right side of (5.108) is the Laplace transform of (5.109). Therefore, we can compute the state transition matrix e^{At} from the Inverse Laplace of $(sI - A)^{-1}$, that is, we can use the relation

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \} \quad (5.110)$$

Next, we consider the output state equation

$$y = Cx + du \quad (5.111)$$

Taking the Laplace of both sides of (5.111), we get

$$Y(s) = CX(s) + dU(s) \quad (5.112)$$

and using (5.108), we get

$$Y(s) = C(sI - A)^{-1} x(0) + [C(sI - A)^{-1} b + d] U(s) \quad (5.113)$$

If the initial condition $x(0) = 0$, (5.113) reduces to

$$Y(s) = [C(sI - A)^{-1} b + d] U(s) \quad (5.114)$$

In (5.114), $U(s)$ is the Laplace transform of the input $u(t)$; then, division of both sides by $U(s)$ yields the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} b + d \quad (5.115)$$

Example 5.12

In the circuit of Figure 5.8, all initial conditions are zero. Compute the state transition matrix e^{At} using the Inverse Laplace transform method.

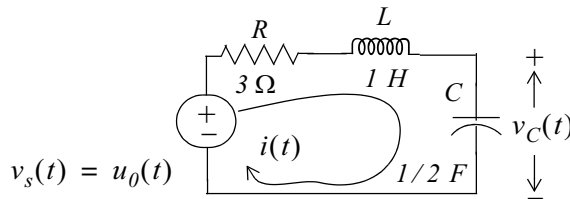


Figure 5.8. Circuit for Example 5.12

Chapter 5 State Variables and State Equations

Solution:

For this circuit,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging,

$$\frac{di_L}{dt} = -3i_L - v_C + 1 \quad (5.116)$$

Now, we define the state variables

$$x_1 = i_L$$

and

$$x_2 = v_C$$

Then,

$$\dot{x}_1 = \frac{di_L}{dt} = -3i_L - v_C + 1 \quad (5.117)$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt} = 0.5\frac{dv_C}{dt} \quad (5.118)$$

and thus,

$$x_1 = i_L = 0.5\frac{dv_C}{dt} = 0.5\dot{x}_2$$

or

$$\dot{x}_2 = 2x_1 \quad (5.119)$$

Therefore, from (5.117) and (5.119) we get the state equations

$$\begin{aligned} \dot{x}_1 &= -3x_1 - x_2 + 1 \\ \dot{x}_2 &= 2x_1 \end{aligned} \quad (5.120)$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \quad (5.121)$$

By inspection,

$$A = \begin{bmatrix} -3 & -I \\ 2 & 0 \end{bmatrix} \quad (5.122)$$

Now, we will find the state transition matrix from

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \} \quad (5.123)$$

where

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -I \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & I \\ -2 & s \end{bmatrix}$$

Then,

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s & -I \\ 2 & s+3 \end{bmatrix} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{-I}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

We find the Inverse Laplace of each term by partial fraction expansion. Then,

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \} = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Now, we can find the state variables representing the inductor current and the capacitor voltage from

$$x(t) = e^{At}x_0 + e^{At} \int_0^t e^{-A\tau} bu(\tau) d\tau$$

using the procedure of Example 5.11.

MATLAB provides two very useful functions to convert state space (state equations), to transfer function (s-domain), and vice versa. The function **ss2tf** (state space to transfer function) converts the state space equations

$$\begin{aligned} \dot{x} &= Ax + Bu * \\ y &= Cx + Du \end{aligned} \quad (5.124)$$

to the rational transfer function form

$$G(s) = \frac{N(s)}{D(s)} \quad (5.125)$$

* We have used capital letters for vectors b and c to be consistent with MATLAB's designations.

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This is used with the statement **[num,den]=ss2tf(A,B,C,D,iu)** where **A, B, C, D** are the matrices of (5.124) and **iu** is 1 if there is only one input. The MATLAB **help** command provides the following information:

help ss2tf

SS2TF State-space to transfer function conversion.
[NUM,DEN] = SS2TF(A,B,C,D,iu) calculates the transfer function:

$$G(s) = \frac{\text{NUM}(s)}{\text{DEN}(s)} = \frac{\text{C}(s\text{I}-\text{A}) \text{B} + \text{D}}{-1}$$

of the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

from the *iu*'th input. Vector DEN contains the coefficients of the denominator in descending powers of *s*. The numerator coefficients are returned in matrix NUM with as many rows as there are outputs *y*.

See also TF2SS

The other function, **tf2ss**, converts the transfer function of (5.125) to the state-space equations of (5.124). It is used with the statement **[A,B,C,D]=tf2ss(num,den)** where **A, B, C**, and **D** are the matrices of (5.124), and **num, den** are *N(s)* and *D(s)* of (5.125) respectively. The MATLAB **help** command provides the following information:

help tf2ss

TF2SS Transfer function to state-space conversion.
[A,B,C,D] = TF2SS(NUM,DEN) calculates the state-space representation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

of the system:

$$G(s) = \frac{\text{NUM}(s)}{\text{DEN}(s)}$$

from a single input. Vector DEN must contain the coefficients of the denominator in descending powers of *s*. Matrix NUM must con-

tain the numerator coefficients with as many rows as there are outputs y . The A, B, C, D matrices are returned in controller canonical form. This calculation also works for discrete systems. To avoid confusion when using this function with discrete systems, always use a numerator polynomial that has been padded with zeros to make it the same length as the denominator. See the User's guide for more details.

See also SS2TF.

Example 5.13

For the circuit of Figure 5.9,

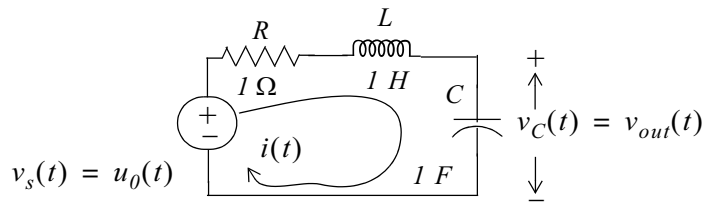


Figure 5.9. Circuit for Example 5.13

a. Derive the state equations and express them in matrix form as

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

b. Derive the transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

c. Verify your answers with MATLAB.

Solution:

a. The differential equation describing the circuit is

$$Ri + L\frac{di}{dt} + v_C = u_0(t)$$

and with the given values,

$$i + \frac{di}{dt} + v_C = u_0(t)$$

or

$$\frac{di}{dt} = -i - v_C + u_0(t)$$

We let

$$x_1 = i_L = i$$

and

$$x_2 = v_C = v_{out}$$

Then,

$$\dot{x}_1 = \frac{di}{dt}$$

and

$$\dot{x}_2 = \frac{dv_c}{dt} = x_1$$

Thus, the state equations are

$$\dot{x}_1 = -x_1 - x_2 + u_0(t)$$

$$\dot{x}_2 = x_1$$

$$y = x_2$$

and in matrix form,

$$\begin{aligned} \dot{x} = Ax + Bu &\leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_0(t) \\ y = Cx + Du &\leftrightarrow y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u_0(t) \end{aligned} \quad (5.126)$$

b. The s – domain circuit is

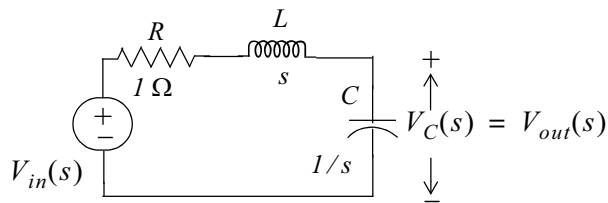


Figure 5.10. Transformed circuit for Example 5.13

By the voltage division expression,

$$V_{out}(s) = \frac{1/s}{1 + s + 1/s} V_{in}(s)$$

or

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{I}{s^2 + s + I}$$

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{I}{s^2 + s + I} \quad (5.127)$$

c.

```
A = [-1 -1; 1 0]; B = [1 0]'; C = [0 1]; D = [0]; % The matrices of (5.126)
[num, den] = ss2tf(A, B, C, D, 1) % Verify coefficients of G(s) in (5.127)

num =
    0    0    1
den =
    1.0000    1.0000    1.0000

num = [0 0 1]; den = [1 1 1]; % The coefficients of G(s) in (5.127)
[A B C D] = tf2ss(num, den) % Verify the matrices of (5.126)

A =
    -1    -1
     1     0
B =
     1
     0
C =
     0     1
D =
     0
```

5.8 Summary

- An n th-order differential equation can be resolved to n first-order simultaneous differential equations with a set of auxiliary variables called state variables. The resulting first-order differential equations are called state space equations, or simply state equations.
- The state space equations can be obtained either from the n th-order differential equation, or directly from the network, provided that the state variables are chosen appropriately.
- When we obtain the state equations directly from given circuits, we choose the state variables to represent inductor currents and capacitor voltages.
- The state variable method offers the advantage that it can also be used with non-linear and time-varying devices.
- If a circuit contains only one energy-storing device, the state equations are written as

$$\begin{aligned}\dot{x} &= \alpha x + \beta u \\ y &= k_1 x + k_2 u\end{aligned}$$

where α , β , k_1 , and k_2 are scalar constants, and the initial condition, if non-zero, is denoted as

$$x_0 = x(t_0) \quad (5.128)$$

- If α and β are scalar constants, the solution of $\dot{x} = \alpha x + \beta u$ with initial condition $x_0 = x(t_0)$ is obtained from the relation

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha\tau} \beta u(\tau) d\tau$$

- The solution of the state equations pair

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= Cx + du\end{aligned}$$

where A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows, entails the computation of the state transition matrix e^{At} , and integration of

$$x(t) = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} bu(\tau) d\tau$$

- The eigenvalues λ_i , where $i = 1, 2, \dots, n$, of an $n \times n$ matrix A are the roots of the n th order polynomial

$$\det[A - \lambda I] = 0$$

where I is the $n \times n$ identity matrix.

- We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix.
- The Cayley-Hamilton theorem states that a matrix can be expressed as an $(n - 1)th$ degree polynomial in terms of the matrix A as

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1}$$

where the coefficients a_i are functions of the eigenvalues λ .

- If all eigenvalues of a given matrix A are distinct, that is, if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \dots \neq \lambda_n$, the coefficients a_i are found from the simultaneous solution of the system of equations

$$a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1} = e^{\lambda_1 t}$$

$$a_0 + a_1 \lambda_2 + a_2 \lambda_2^2 + \dots + a_{n-1} \lambda_2^{n-1} = e^{\lambda_2 t}$$

...

$$a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \dots + a_{n-1} \lambda_n^{n-1} = e^{\lambda_n t}$$

- If some or all eigenvalues of matrix A are repeated, that is, if $\lambda_1 = \lambda_2 = \lambda_3 \dots = \lambda_m$, λ_{m+1} , λ_n , the coefficients a_i of the state transition matrix are found from the simultaneous solution of the system of equations

$$a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1} = e^{\lambda_1 t}$$

$$\frac{d}{d\lambda_1} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) = \frac{d}{d\lambda_1} e^{\lambda_1 t}$$

$$\frac{d^2}{d\lambda_1^2} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) = \frac{d^2}{d\lambda_1^2} e^{\lambda_1 t}$$

...

$$\frac{d^{m-1}}{d\lambda_1^{m-1}} (a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 + \dots + a_{n-1} \lambda_1^{n-1}) = \frac{d^{m-1}}{d\lambda_1^{m-1}} e^{\lambda_1 t}$$

$$a_0 + a_1 \lambda_{m+1} + a_2 \lambda_{m+1}^2 + \dots + a_{n-1} \lambda_{m+1}^{n-1} = e^{\lambda_{m+1} t}$$

...

$$a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \dots + a_{n-1} \lambda_n^{n-1} = e^{\lambda_n t}$$

- A column vector X that satisfies the relation

$$AX = \lambda X$$

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where A is an $n \times n$ matrix and λ is a scalar number is called an eigenvector.

$$\det(A - \lambda I) = 0 \quad (5.129)$$

- There is a different eigenvector for each eigenvalue.
- Eigenvectors are generally expressed as unit eigenvectors, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.
- Two vectors X and Y are said to be *orthogonal* if their inner (dot) product is zero.
- A set of eigenvectors constitutes an orthonormal basis if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal.
- The state transition matrix can be computed from the Inverse Laplace transform using the relation

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

- If $U(s)$ is the Laplace transform of the input $u(t)$ and $Y(s)$ is the Laplace transform of the output $y(t)$, the transfer function can be computed using the relation

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

- MATLAB provides two very useful functions to convert state space (state equations), to transfer function (s-domain), and vice versa. The function **ss2tf** (state space to transfer function) converts the state space equations to the transfer function equivalent, and the function **tf2ss**, converts the transfer function to state-space equations.

5.9 Exercises

1. Express the integrodifferential equation below as a matrix of state equations where k_1 , k_2 , and k_3 are constants.

$$\frac{dv^2}{dt^2} + k_3 \frac{dv}{dt} + k_2 v + k_1 \int_0^t v dt = \sin 3t + \cos 3t$$

2. Express the matrix of the state equations below as a single differential equation, and let $x(y) = y(t)$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

3. For the circuit of Figure 5.11, all initial conditions are zero, and $u(t)$ is any input. Write state equations in matrix form.

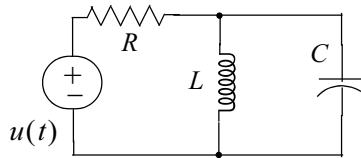


Figure 5.11. Circuit for Exercise 3

4. In the circuit of Figure 5.12, all initial conditions are zero. Write state equations in matrix form.

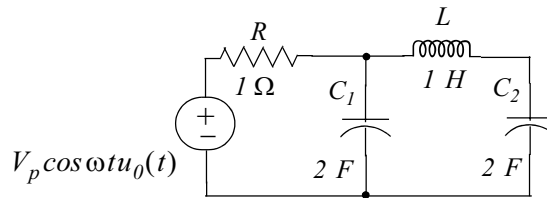


Figure 5.12. Circuit for Exercise 4

5. In the circuit of Figure 5.13, $i_L(0^-) = 2 \text{ A}$. Use the state variable method to find $i_L(t)$ for $t > 0$.

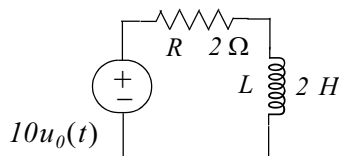


Figure 5.13. Circuit for Exercise 5

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6. Compute the eigenvalues of the matrices A , B , and C below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Hint: One of the eigenvalues of matrix C is -1 .

7. Compute e^{At} given that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Observe that this is the same matrix as C of Exercise 6.

8. Find the solution of the matrix state equation $\dot{x} = Ax + bu$ given that

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u = \delta(t), \quad t_0 = 0$$

9. In the circuit of Figure 5.14, $i_L(0^-) = 0$, and $v_C(0^-) = 1$ V.

- Write state equations in matrix form.
- Compute e^{At} using the Inverse Laplace transform method.
- Find $i_L(t)$ and $v_C(t)$ for $t > 0$.

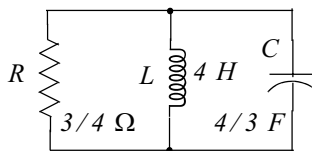


Figure 5.14. Circuit for Exercise 9

5.10 Solutions to Exercises

1. Differentiating the given integrodifferential equation with respect to t we get

$$\frac{dv^3}{dt^3} + k_3 \frac{dv^2}{dt^2} + k_2 \frac{dv}{dt} + k_1 v = 3 \cos 3t - 3 \sin 3t = 3(\cos 3t - \sin 3t)$$

or

$$\frac{dv^3}{dt^3} = -k_3 \frac{dv^2}{dt^2} - k_2 \frac{dv}{dt} - k_1 v + 3(\cos 3t - \sin 3t) \quad (1)$$

We let

$$v = x_1 \quad \frac{dv}{dt} = x_2 = \dot{x}_1 \quad \frac{dv^2}{dt^2} = x_3 = \dot{x}_2$$

Then,

$$\frac{dv^3}{dt^3} = \dot{x}_3$$

and by substitution into (1)

$$\dot{x}_3 = -k_1 x_1 - k_2 x_2 - k_3 x_3 + 3(\cos 3t - \sin 3t)$$

and thus the state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -k_1 x_1 - k_2 x_2 - k_3 x_3 + 3(\cos 3t - \sin 3t)$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot 3(\cos 3t - \sin 3t)$$

2.

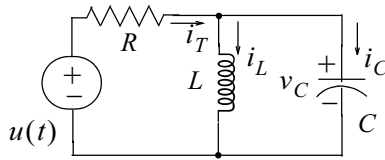
Expansion of the given matrix yields

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = x_4 \quad \dot{x}_4 = -x_1 - 2x_2 - 3x_3 - 4x_4 + u(t)$$

Letting $x = y$ we get

$$\frac{dy^4}{dt^4} + 4 \frac{dy^3}{dt^3} + 3 \frac{dy^2}{dt^2} + 2 \frac{dy}{dt} + y = u(t)$$

3.



We let $i_L = x_1$ and $v_C = x_2$. By KCL, $i_T = i_L + i_C$ or

$$\frac{u(t) - v_C}{R} = i_L + C \frac{dv_C}{dt}$$

or

$$\frac{u(t) - x_2}{R} = x_1 + C \dot{x}_2$$

Also,

$$x_2 = L \dot{x}_1$$

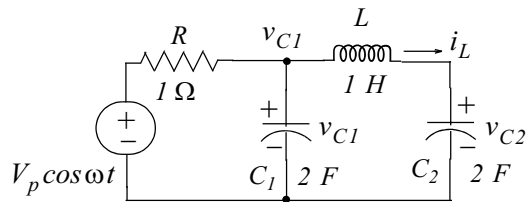
Then,

$$\dot{x}_1 = \frac{1}{L} x_2 \text{ and } \dot{x}_2 = -\frac{1}{C} x_1 - \frac{1}{RC} x_2 + \frac{1}{RC} u(t)$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} \cdot u(t)$$

4.



We let $i_L = x_1$, $v_{C1} = x_2$, and $v_{C2} = x_3$. By KCL

$$\frac{v_{C1} - V_p \cos \omega t}{1} + 2 \frac{dv_{C1}}{dt} + i_L = 0 \text{ or } x_2 - V_p \cos \omega t + 2 \dot{x}_2 + x_1 = 0$$

or

$$\dot{x}_2 = -\frac{1}{2} x_1 - \frac{1}{2} x_2 + \frac{1}{2} V_p \cos \omega t \quad (1)$$

By KVL

$$v_{C1} = L \frac{di_L}{dt} + v_{C2} \text{ or } x_2 = L\dot{x}_1 + x_3 \text{ or } \dot{x}_1 = x_2 - x_3 \quad (2)$$

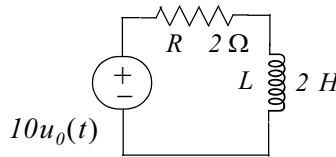
Also,

$$i_L = C \frac{dv_{C2}}{dt} \text{ or } x_1 = 2\dot{x}_3 \text{ or } \dot{x}_3 = \frac{1}{2}x_1 \quad (3)$$

Combining (1), (2), and (3) into matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1/2 & -1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} \cdot V_p \cos \omega t$$

5.



From (5.21) of Example 5.4

$$\dot{x} = -\frac{R}{L}x + \frac{1}{L}v_S u_0(t)$$

For this exercise $\alpha = -R/L = -1$ and $b = 10 \times (1/L) = 5$. Then,

$$\begin{aligned} x(t) &= e^{\alpha(t-t_0)} x_0 + e^{\alpha t} \int_{t_0}^t e^{-\alpha \tau} \beta u(\tau) d\tau \\ &= e^{-1(t-0)} 2 + e^{-t} \int_0^t e^{\tau} 5u_0(\tau) d\tau = 2e^{-t} + 5e^{-t} \int_0^t e^{\tau} d\tau \\ &= 2e^{-t} + 5e^{-t}(e^t - 1) = 2e^{-t} + 5 - 5e^{-t} = (5 - 3e^{-t})u_0(t) \end{aligned}$$

and denoting the current i_L as the output y we get

$$y(t) = x(t) = (5 - 3e^{-t})u_0(t)$$

6.

a.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 6 = 0$$

$$-1 - \lambda + \lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 = 7$$

and thus

$$\lambda_1 = \sqrt{7} \quad \lambda_2 = -\sqrt{7}$$

b.

$$B = \begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} \quad \det(B - \lambda I) = \det\left(\begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} a - \lambda & 0 \\ -a & b - \lambda \end{bmatrix} = 0$$

$$(a - \lambda)(b - \lambda) = 0$$

and thus

$$\lambda_1 = a \quad \lambda_2 = b$$

c.

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \det(C - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6 - \lambda \end{bmatrix} = 0$$

$$\lambda^2(-6 - \lambda) - 6 - (-11)(-\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

and it is given that $\lambda_1 = -1$. Then,

$$\frac{\lambda^3 + 6\lambda^2 + 11\lambda + 6}{(\lambda + 1)} = \lambda^2 + 5\lambda + 6 \Rightarrow (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

and thus

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

7.

a. Matrix A is the same as Matrix C in Exercise 6. Then,

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

and since A is a 3×3 matrix the state transition matrix is

$$e^{At} = a_0 I + a_1 A + a_2 A^2 \quad (1)$$

Then,

$$a_0 + a_1 \lambda_1 + a_2 \lambda_1^2 = e^{\lambda_1 t} \Rightarrow a_0 - a_1 + a_2 = e^{-t}$$

$$a_0 + a_1 \lambda_2 + a_2 \lambda_2^2 = e^{\lambda_2 t} \Rightarrow a_0 - 2a_1 + 4a_2 = e^{-2t}$$

$$a_0 + a_1 \lambda_3 + a_2 \lambda_3^2 = e^{\lambda_3 t} \Rightarrow a_0 - 3a_1 + 9a_2 = e^{-3t}$$

```
syms t; A=[1 -1 1; 1 -2 4; 1 -3 9];...
a=sym('[exp(-t); exp(-2*t); exp(-3*t)]'); x=A\ a; fprintf(' \n');...
disp('a0 = '); disp(x(1)); disp('a1 = '); disp(x(2)); disp('a2 = '); disp(x(3))
```

```
a0 =
3*exp(-t) - 3*exp(-2*t) + exp(-3*t)
```

```
a1 =
5/2*exp(-t) - 4*exp(-2*t) + 3/2*exp(-3*t)
```

```
a2 =
1/2*exp(-t) - exp(-2*t) + 1/2*exp(-3*t)
```

Thus,

$$a_0 = 3e^{-t} - 3e^{-2t} + 3e^{-3t}$$

$$a_1 = 2.5e^{-t} - 4e^{-2t} + 1.5e^{-3t}$$

$$a_2 = 0.5e^{-t} - e^{-2t} + 0.5e^{-3t}$$

Now, we compute e^{At} of (1) with the following MATLAB code:

```
syms t; a0=3*exp(-t)-3*exp(-2*t)+exp(-3*t); a1=5/2*exp(-t)-4*exp(-2*t)+3/2*exp(-3*t);...
a2=1/2*exp(-t)-exp(-2*t)+1/2*exp(-3*t); A=[0 1 0; 0 0 1; -6 -11 -6]; fprintf(' \n');...
eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
```

```
[ 3*exp(-t) - 3*exp(-2*t) + exp(-3*t), 5/2*exp(-t) - 4*exp(-2*t) + 3/2*exp(-3*t), 1/2*exp(-t) - exp(-2*t) + 1/2*exp(-3*t) ]
[-3*exp(-t) + 6*exp(-2*t) - 3*exp(-3*t), -5/2*exp(-t) + 8*exp(-2*t) - 9/2*exp(-3*t), -1/2*exp(-t) + 2*exp(-2*t) - 3/2*exp(-3*t) ]
[3*exp(-t) - 12*exp(-2*t) + 9*exp(-3*t), 5/2*exp(-t) - 16*exp(-2*t) + 27/2*exp(-3*t), 1/2*exp(-t) - 4*exp(-2*t) + 9/2*exp(-3*t) ]
```

Then,

$$e^{At} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} + e^{-3t} & 2.5e^{-t} - 4e^{-2t} + 1.5e^{-3t} & 0.5e^{-t} - e^{-2t} + 0.5e^{-3t} \\ -3e^{-t} + 6e^{-2t} - 3e^{-3t} & -2.5e^{-t} + 8e^{-2t} - 4.5e^{-3t} & -0.5e^{-t} + 2e^{-2t} - 1.5e^{-3t} \\ 3e^{-t} - 12e^{-2t} + 9e^{-3t} & 2.5e^{-t} - 16e^{-2t} + 13.5e^{-3t} & 0.5e^{-t} - 4e^{-2t} + 4.5e^{-3t} \end{bmatrix}$$

8.

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u = \delta(t), \quad t_0 = 0$$

$$\begin{aligned}
 x(t) &= e^{A(t-0)}x_0 + e^{At} \int_0^t e^{-A\tau} bu(\tau) d\tau = e^{At}x_0 + e^{At} \int_0^t e^{-A\tau} b\delta(\tau) d\tau \\
 &= e^{At}x_0 + e^{At}b = e^{At}(x_0 + b) = e^{At} \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = e^{At} \begin{bmatrix} 0 \\ 2 \end{bmatrix}
 \end{aligned} \tag{1}$$

We use the following MATLAB code to find the eigenvalues λ_1 and λ_2

```
A=[1 0; -2 2]; lambda=eig(A); fprintf(' \n');...
fprintf('lambda1 = %4.2f \t',lambda(1)); fprintf('lambda2 = %4.2f \t',lambda(2))
```

```
lambda1 = 2.00    lambda2 = 1.00
```

Next,

$$\begin{aligned}
 a_0 + a_1\lambda_1 &= e^{\lambda_1 t} \Rightarrow a_0 + a_1 = e^t \\
 a_0 + a_1\lambda_2 &= e^{\lambda_2 t} \Rightarrow a_0 + 2a_1 = e^{2t}
 \end{aligned}$$

Then,

$$a_0 = 2e^t - e^{2t} \quad a_1 = e^{2t} - e^t$$

and

$$\begin{aligned}
 e^{At} &= a_0I + a_1A = (2e^t - e^{2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{2t} - e^t) \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 2e^t - e^{2t} & 0 \\ 0 & 2e^t - e^{2t} \end{bmatrix} + \begin{bmatrix} e^{2t} - e^t & 0 \\ -2e^{2t} + 2e^t & 2e^{2t} - 2e^t \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 2e^t - 2e^{2t} & e^{2t} \end{bmatrix}
 \end{aligned}$$

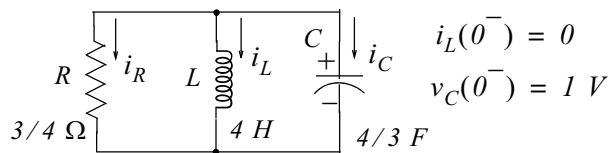
By substitution into (1) we get

$$x(t) = e^{At} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 2e^t - 2e^{2t} & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix}$$

and thus

$$x_1 = 0 \quad x_2 = 2e^{2t}$$

9.



We let $x_1 = i_L$ $x_2 = v_C$. Then,

a.

$$i_R + i_L + i_C = 0$$

$$\frac{v_C}{R} + i_L + C \frac{v_C}{dt} = 0$$

$$\frac{x_2}{3/4} + x_1 + \frac{4}{3} \dot{x}_2 = 0$$

or

$$\dot{x}_2 = -\frac{3}{4}x_1 - x_2 \quad (1)$$

Also,

$$v_L = v_C = L \frac{di_L}{dt} = 4\dot{x}_1 = x_2$$

or

$$\dot{x}_1 = \frac{1}{4}x_2 \quad (2)$$

From (1) and (2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and thus

$$A = \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix}$$

b.

$$e^{At} = \mathcal{L}^{-1} \{ [sI - A]^{-1} \}$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix} = \begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix}$$

$$\Delta = \det[sI - A] = \det \begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix} = s^2 + s + 3/16 = (s + 1/4)(s + 3/4)$$

$$\text{adj}[sI - A] = \text{adj} \begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix} = \begin{bmatrix} s+1 & 1/4 \\ -3/4 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{\Delta} \text{adj}[sI - A] = \frac{1}{(s + 1/4)(s + 3/4)} \begin{bmatrix} s + 1 & 1/4 \\ -3/4 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s + 1}{(s + 1/4)(s + 3/4)} & \frac{1/4}{(s + 1/4)(s + 3/4)} \\ \frac{-3/4}{(s + 1/4)(s + 3/4)} & \frac{s}{(s + 1/4)(s + 3/4)} \end{bmatrix}$$

We use MATLAB to find $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$ with the code below.

```
syms s t
Fs1=(s+1)/(s^2+s+3/16); Fs2=(1/4)/(s^2+s+3/16); Fs3=(-3/4)/(s^2+s+3/16); Fs4=s/(s^2+s+3/16);...
fprintf(' \n'); disp('a11 = '); disp(simple(ilaplace(Fs1))); disp('a12 = '); disp(simple(ilaplace(Fs2)));...
disp('a21 = '); disp(simple(ilaplace(Fs3))); disp('a22 = '); disp(simple(ilaplace(Fs4)))

a11 =
-1/2*exp(-3/4*t)+3/2*exp(-1/4*t)

a12 =
1/2*exp(-1/4*t)-1/2*exp(-3/4*t)

a21 =
-3/2*exp(-1/4*t)+3/2*exp(-3/4*t)

a22 =
3/2*exp(-3/4*t)-1/2*exp(-1/4*t)
```

Thus,

$$e^{At} = \begin{bmatrix} 1.5e^{-0.25t} - 0.5e^{-0.75t} & 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -1.5e^{-0.25t} + 1.5e^{-0.75t} & -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix}$$

c.

$$x(t) = e^{A(t-0)}x_0 + e^{At} \int_0^t e^{-A\tau} bu(\tau) d\tau = e^{At}x_0 + 0 = e^{At} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1.5e^{-0.25t} - 0.5e^{-0.75t} & 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -1.5e^{-0.25t} + 1.5e^{-0.75t} & -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix}$$

and thus for $t > 0$

$$x_1 = i_L = 0.5e^{-0.25t} - 0.5e^{-0.75t} \quad x_2 = v_C = -0.5e^{-0.25t} + 1.5e^{-0.75t}$$